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Hawking radiation of charged particles via tunneling from the Reissner-Nordström black hole

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Abstract: Hawking radiation is viewed as a tunneling process. In this way the emission rate at which charged massive particles tunnel across the event horizon of the Reissner-Nordström black hole is calculated. The emission spectrum deviates from the pure thermal spectrum, but it is consistent with an underlying unitary theory and takes the same functional form as that of uncharged massless particles.

Keywords: Black Holes, Classical Theories of Gravity.

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Contents

1. Introduction

In 2000, Parikh and Wilczek proposed a method to calculate the emission rate at which particles tunnel across the event horizon [1 − 3]. They treat Hawking radiation as a tunneling process. They find that the barrier is created by the outgoing particle itself, and their Key insight is to find a coordinate system well-behaved at the event horizon to calculate the emission rate. In this way they have calculated the corrected emission spectrum of the uncharged particles from the spherically symmetric black holes, such as Schwarzshild black holes and Reissner-Norström black holes. Their results are consistent with an underlying unitary theory. But this approach is only suitable for us to calculate the emission rate of the uncharged massless particles. In this paper, we attempt to extend this method to calculate the charged massive particle's emission rate. There are two difficulties to overcome. The first is that in order to do a computation, one need to find the equation of motion of a charged massive tunneling particle. The second is how to take into account the effect of the electromagnetic field outside the hole. In the following discussion, we first investigate the behavior of a charged massive tunneling particle, and obtain the phase velocity and group velocity of the de Broglie wave corresponding to the outgoing particle. Then, we discuss the effect of the electromagnetic field. We find that the lagrangian density of the electromagnetic field can be expressed by a set of generalized coordinates $A_\mu = (A_t, A_1, A_2, A_3)$. But these coordinates are ignorable coordinates. To eliminate the freedoms corresponding to these coordinates, we modify the lagrangian function and use Parikh's method to get the corrected emission spectrum. Finally, we give a short discussion about our result. Throughout the paper, the geometrized units ($G \equiv c \equiv \hbar \equiv 1$) are used.

2. Tunneling rate

The Painleve line element of the Ressiner-Norström black hole is

$$ds^2 = -\left(1 - \frac{2M}{r} + \frac{Q^2}{r^2}\right) dt^2 + 2\sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}} dtdr + dr^2 + r^2 d\Omega^2 \,, \tag{2.1}$$

which is obtained from the standard Reissner-Nordström line element by the coordinate transformation[1],

$$t = t_r + 2\sqrt{2Mr - Q^2} + M \ln\left(\frac{r - \sqrt{2Mr - Q^2}}{r + \sqrt{2Mr - Q^2}}\right) + \frac{Q^2 - M^2}{\sqrt{M^2 - Q^2}} \times \\ \times \arctan\left(\frac{\sqrt{M^2 - Q^2}\sqrt{2Mr - Q^2}}{Mr}\right), \tag{2.2}$$

where t_r is the Reissner time coordinate.

As is described in ref. [1], line element (2.1) displays the stationary, nonstatic, and nonsingular nature of the spacetime. Moreover, we find another important feature which is rarely stressed in the relevant literatures. We describe this as follows.

According to Landau's theory of the coordinate clock synchronization [4] in a spacetime decomposed in (3+1), the coordinate time difference of two events, which take place simultaneously in different places, is

$$\Delta T = -\int \frac{g_{0i}}{g_{00}} dx^i \qquad (i = 1, 2, 3)\,. \tag{2.3}$$

If the simultaneity of coordinate clocks can be transmitted from one place to another and has nothing to do with the integration path, components of the metric should satisfy [5]

$$\frac{\partial}{\partial x^j}\left(-\frac{g_{0i}}{g_{00}}\right) = \frac{\partial}{\partial x^i}\left(-\frac{g_{0j}}{g_{00}}\right) \qquad (i, j = 1, 2, 3)\,. \tag{2.4}$$

Obviously, the line element (2.1) satisfies condition (2.4), that is, the coordinate clock synchronization in the Painleve coordinates can be transmitted from one place to another though the line element is not diagonal. In quantum mechanics, particle tunneling across a barrier is an instantaneous process. Thus, this feature is necessary for us to discuss the tunneling process.

From the coordinate transformation (2.2), we get the 4-dimensional electromagnetic potential

$$A_\mu = (A_t, 0, 0, 0)\,, \tag{2.5}$$

where $A_t = -Q/r$. The radial null geodesics are given by[1]

$$\dot{r} \equiv \frac{dr}{dt} = \pm 1 - \sqrt{\frac{2M}{r} - \frac{Q^2}{r^2}}\,, \tag{2.6}$$

with the upper(lower) sign in eq. (2.6) corresponding to outgoing(ingoing) geodesics, under the implicit assumption that t increases towards the future.

But in this paper we consider the tunneling of charged massive particles. That is, the outgoing particle is a massive shell (de Broglie s-wave). The charged massive quanta doesn't follow radial-lightlike geodesics (2.6). According to de Broglie hypothesis, this massive shell is a sort of de Broglie s-wave. The approximative wave equation is

$$\psi(r, t) = Ce^{i(\int_{r_i - \varepsilon}^{r} p_r dr - \omega t)}\,, \tag{2.7}$$

where $r_i - \varepsilon$ represents the initial location of the particle. If we let

$$\int_{r_i-\varepsilon}^{r} p_r dr - \omega t = \phi_0 \,, \tag{2.8}$$

then, we have

$$\frac{dr}{dt} = \dot{r} = \frac{\omega}{k} \,, \tag{2.9}$$

where k is the de Broglie wave number. Comparing eq. (2.9) with the definition of the phase velocity we know that $\dot{r}$ is the phase velocity of the de Broglie wave. Unlike the electromagnetic wave, the phase velocity v_p of the de Broglie wave is not equal to the group velocity v_g. The definitions and relationship between them are

$$\begin{aligned} v_p &= \frac{dr}{dt} = \dot{r} = \frac{\omega}{k} \,, \\ v_g &= \frac{dr_c}{dt} = \frac{d\omega}{dk} , \\ v_p &= \frac{1}{2} v_g \,. \end{aligned} \tag{2.10}$$

To obtain the formula of the phase velocity $\dot{r}$, let's now investigate the behavior of a massive particle tunneling across the event horizon.

Since tunneling across the barrier is an instantaneous process, there are two simultaneous events during the process. one is particle tunneling into the barrier, another is particle tunneling out the barrier. In terms of Landau's theory of the coordinate clock synchronization, the difference of coordinate times of these two simultaneous events is

$$dt = -\frac{g_{0i}}{g_{00}} dx^i = -\frac{g_{01}}{g_{00}} dr_c \,, \qquad (d\theta = d\varphi = 0) \,, \tag{2.11}$$

where r_c denotes the location of the tunnneling particle. The group velocity is

$$v_g = \frac{dr_c}{dt} = -\frac{g_{00}}{g_{01}} \,, \tag{2.12}$$

and the phase velocity is therefore

$$\dot{r} = v_p = \frac{1}{2} v_g = -\frac{1}{2} \frac{g_{00}}{g_{01}} = \frac{1}{2r} \frac{r^2 - 2Mr + Q^2}{\sqrt{2Mr - Q^2}} \,. \tag{2.13}$$

Moreover, if the self-gravitation is included, eqs. (2.1), (2.12) and (2.13) should be replaced by $M \to M - \omega$, $Q \to Q - q$, where ω and q are particle's energy and electric charge, respectively.

When we investigate a charged particle's tunneling process, the effect of the electromagnetic field should be taken into account. That is, the matter-gravity system consists of the black hole and the electromagnetic field outside the hole. We write the lagrangian function of the matter-gravity system as

$$L = L_m + L_e \,, \tag{2.14}$$

JHEP10(2005)055

where $L_e = -1/4F_{\mu\nu}F^{\mu\nu}$ is the lagrangian function of the electromagnetic field corresponding to the generalized coordinates $A_\mu = (A_t, 0, 0, 0)$ [6]. When a charged particle tunnels out, the system transit from one state to another. But from the expression of L_e we find that $A_\mu = (A_t, 0, 0, 0)$ is an ignorable coordinate. To eliminate the freedom corresponding to A_t, the action should be written as

$$S = \int_{t_i}^{t_f} (L - P_{A_t}\dot{A}_t)dt\,, \tag{2.15}$$

which is related to the emission rate of the tunneling particle by

$$\Gamma \sim e^{-2\,\mathrm{Im}\,S}\,. \tag{2.16}$$

Therefore, the imaginary part of the action is

$$\mathrm{Im}\,S = \mathrm{Im}\left\{\int_{r_i}^{r_f}\left[P_r - \frac{P_{A_t}\dot{A}_t}{\dot{r}}\right]dr\right\} = \mathrm{Im}\left\{\int_{r_i}^{r_f}\left[\int_{(0,0)}^{(P_r,P_{A_t})} dP_r' - \frac{\dot{A}_t}{\dot{r}}dP_{A_t}'\right]dr\right\}, \tag{2.17}$$

where P_{A_t} is the electromagnetic field's canonical momentum conjugate to A_t.

To proceed with an explicit calculation, it is useful to apply Hamilton's equations

$$\dot{r} = \frac{dH}{dP_r}\,|_{(r;A_t,P_{A_t})}, \tag{2.18}$$

$$\dot{A}_t = \frac{dH}{dP_{A_t}}\,|_{(A_t;r,P_r)}\,. \tag{2.19}$$

Substituting eqs. (2.18) and (2.19) into eq. (2.17) yields

$$\mathrm{Im}\,S = \mathrm{Im}\left\{\int_{r_i}^{r_f}\left[\int_{(M,E_Q)}^{(M-\omega,E_{Q-q})} \frac{1}{\dot{r}}(dH)_{r;A_t,P_{A_t}} - \frac{1}{\dot{r}}(dH)_{A_t;r,P_r}\right]dr\right\}. \tag{2.20}$$

Here, E_Q represents the energy of the electromagnetic field, and

$$(dH)_{r;A_t,P_{A_t}} = d(M - \omega') = -d\omega'\,, \tag{2.21}$$

$$(dH)_{A_t;r,P_r} = -\frac{Q - q'}{r}dq'\,. \tag{2.22}$$

Eqs. (2.21) and (2.22) represent energy changes of hole because of the loss of mass and charge when a particle tunnels out. In eq. (2.22), we have treated the black hole as a charged conducting sphere [7].

Based on similar discussion to ref. [1] and [2], it follows directly that a charged particle tunneling across the event horizon sees the effective metric of eq. (2.1), although with the replaces $M \to M - \omega'$ and $Q \to Q - q'$. The same substitution in eq. (2.13) yields the desired expression for $\dot{r}$ as a function of ω' and q'. Thus, we can rewrite eq. (2.20) in the following explicit manner

$$\operatorname{Im} S = -\operatorname{Im} \int_{r_i}^{r_f} \left[\int_{(0,0)}^{(\omega,q)} \frac{2r\sqrt{2(M-\omega')r-(Q-q')^2}}{(r-r'_+)(r-r'_-)} d\omega' - \right. \\ \left. - \frac{(Q-q')2\sqrt{2(M-\omega')r-(Q-q')^2}}{(r-r'_+)(r-r'_-)} dq' \right] dr \,, \tag{2.23}$$

where

$$r'_\pm = M-\omega' \pm \sqrt{(M-\omega')^2-(Q-q')^2} \,, \tag{2.24}$$
$$r_i = M+\sqrt{M^2-Q^2} \,, \tag{2.25}$$
$$r_f = M-\omega+\sqrt{(M-\omega)^2-(Q-q)^2} \,. \tag{2.26}$$

Switching the order of integration and doing the r integral first. We see that $r = r'_+ = M-\omega'+\sqrt{(M-\omega')^2-(Q-q')^2}$ is a pole. The integral can be evaluated by deforming the contour around the pole. In this way we obtain

$$\operatorname{Im} S = -\pi \left[\int_{(0,0)}^{(\omega,q)} \frac{2(r'_+)^2}{r'_+-r'_-} d\omega' - \frac{2r'_+(Q-q')}{r'_+-r'_-} dq' \right] . \tag{2.27}$$

Finishing above integral yields

$$\operatorname{Im} S = -\frac{\pi}{2} \left\{ \left[(M-\omega)+\sqrt{(M-\omega)^2-(Q-q)^2} \right]^2 - \left[M+\sqrt{M^2-Q^2} \right]^2 \right\} = -\frac{1}{2}\Delta S_{BH}, \tag{2.28}$$

where $\Delta S_{BH} = S_{BH}(M-\omega, Q-q) - S_{BH}(M,Q)$ is the difference of the entropies of the black hole before and after the emission.

The tunneling rate is therefore

$$\Gamma \sim \exp[-2\operatorname{Im} S] = e^{\Delta S_{BH}} \,. \tag{2.29}$$

3. Discussion

The emission spectrum (2.29) is consistent with an underlying unitary theory, and it takes the same functional form as that of uncharged massless particles [1]. Moreover, eq. (2.29) also implies that the spectrum is not precisely thermal. To compare with the pure thermal spectrum, we Expand ΔS_{BH} in ω and q,

$$\Delta S_{BH} = \frac{-2\pi(M+\sqrt{M^2-Q^2})^2}{\sqrt{M^2-Q^2}} \times \\ \times \left\{ \omega - \frac{Q}{M+\sqrt{M^2-Q^2}} q - \frac{2(M+\sqrt{M^2-Q^2})(M^2-Q^2)-MQ^2}{2(M^2-Q^2)(M+\sqrt{M^2-Q^2})^2}\omega^2 - \right. \\ - \frac{Q^3}{(M^2-Q^2)(M+\sqrt{M^2-Q^2})^2}\omega q + \\ \left. + \frac{M^3+(M^2-Q^2)\sqrt{M^2-Q^2}}{2(M^2-Q^2)(M+\sqrt{M^2-Q^2})^2} q^2 + O(\omega,q) \right\} . \tag{3.1}$$

If the quadratic term and higher-order terms of ω and q are neglected, we have

$$\Delta S_{BH} = \frac{-2\pi(M+\sqrt{M^2-Q^2})^2}{\sqrt{M^2-Q^2}}\left(\omega - \frac{Q}{M+\sqrt{M^2-Q^2}}q\right) = -\beta(\omega-\omega_0). \qquad (3.2)$$

Namely,

$$\Gamma \sim e^{-\beta(\omega-\omega_0)}\,, \qquad (3.3)$$

where $\omega_0 = \frac{Qq}{M+\sqrt{M^2-Q^2}}$, and $\beta = \frac{2\pi(M+\sqrt{M^2-Q^2})^2}{\sqrt{M^2-Q^2}}$ is the inverse temperature. That is, the corrected spectrum is not precisely thermal. Only the leading-order term givers the thermal Boltzmann factor $e^{-\beta(\omega-\omega_0)}$.

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